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REPRESENTATIONS OF THE q-DEFORMED ALGEBRA $U'_q(so_{2,2})$

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Abstract

The main aim of this paper is to give classes of irreducible infinite dimensional representations and of irreducible ∗-representations of the q-deformed algebra $U'_q(so_{2,2})$ which is a real form of the non-standard deformation $U'_q(so_4)$ of the universal enveloping algebra $U(so(4, \mathbf{C}))$. These representations are described by two complex parameters and are obtained by "analytical continuation" of the irreducible finite dimensional representations of the algebra $U'_q(so_4)$ in the basis corresponding to the reduction from $U'_q(so_4)$ to $U(so_2 \oplus so_2)$.

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1. Introduction

In the classical case, the embedding $SO(n) \subset U(n)$ is of great importance in the theory of Riemannian spaces and for group theoretical approach to some physical problems. In the frame of Drinfeld–Jimbo quantum groups we cannot construct the corresponding embeddings. In this frame we also cannot construct the quantum algebras $U_q(\mathrm{so}_{n,1})$ and introduce Gel'fand–Tsetlin bases in spaces of irreducible representations of $U_q(\mathrm{so}_n)$. To remove these defects the new q-deformation of the universal enveloping algebra $U(\mathrm{so}(n, \mathbf{C}))$ was defined in [1] (see also [2]). We denote it by $U_q'(\mathrm{so}_n)$. This q-deformed algebra allows the embedding $U_q'(\mathrm{so}_{n-1}) \subset U_q'(\mathrm{so}_n)$ and, therefore, we can introduce Gel'fand–Tsetlin bases. It was shown by Noumi [3] that this algebra can be embedded into $U_q(\mathrm{sl}_n)$. The last fact make the algebra $U_q'(\mathrm{so}_n)$ very attractive since the pair $U_q'(\mathrm{so}_n) \subset U_q(\mathrm{sl}_n)$ is of great importance in mathematics and theoretical physics. Of course, for applications we must have good developed representation theory of $U_q'(\mathrm{so}_n)$. Representations of this algebra for q not a root of unity where studied in several papers (see [1, 2]). For q a root of unity only representations of the algebra $U_q'(\mathrm{so}_3)$ were investigated [4, 5].

The algebra $U_q'(\mathrm{so}_n)$ has real forms $U_q'(\mathrm{so}_{m,r})$. The representation theory of the algebras $U_q'(\mathrm{so}_{2,1})$ and $U_q'(\mathrm{so}_{3,1})$ is developed in [2]. The aim of this paper is to construct irreducible representations of $U_q'(\mathrm{so}_{2,2})$ when q is a positive number. In order to construct representations of the algebra $U_q'(\mathrm{so}_{2,2})$ we derive formulas for irreducible finite dimensional representations of $U_q'(\mathrm{so}_4)$ (see [1] and [6]) in the basis corresponding to the reduction from $U_q'(\mathrm{so}_4)$ to $U(\mathrm{so}_2 \oplus \mathrm{so}_2)$. Then we "analytically continue" these formulas to obtain infinite dimensional representations of $U_q'(\mathrm{so}_{2,2})$. In this way we obtain the series of representations T_{bc}^{ϵ} given by two complex parameters. Theorem 1 below describe when these representations are irreducible. Then we study reducible representations T_{bc}^{ϵ} and separate irreducible constituents from them. In this way we obtain Theorem 2 describing all irreducible representations which can be obtained from the representations T_{bc}^{ϵ}. Then in Theorem 3 we separate all $*$-representations from the set of irreducible representations of Theorem 2. Comparing Theorems 2 and 3 with the representation theory of the Lie algebra $\mathrm{so}_{2,2}$, we see that there exists big difference between representation theories of $U_q'(\mathrm{so}_{2,2})$ and $\mathrm{so}_{2,2}$. The same can be said about representation theories of $U_q'(\mathrm{so}_{2,2})$ and $U_q'(\mathrm{so}_{3,1})$. These differences are emphasized in Conclusions at the end of the paper.

2. The algebra $U_q'(\mathrm{so}_{2,2})$

Drinfeld [7] and Jimbo [8] defined q-deformed (quantum) algebras $U_q(g)$ for all simple complex Lie algebras g by means of Cartan subalgebras and root subspaces (see also [9]). However, these approaches do not give a satisfactory presentation of the quantum algebra $U_q(\mathrm{so}_n)$ from point of view of some problems of quantum physics and representation theory. In fact, they admit the inclusion $U_q(\mathrm{so}_n) \supset U_q(\mathrm{so}_{n-2})$ and do not admit the inclusion

$$U_q(\mathrm{so}_n) \supset U_q(\mathrm{so}_{n-1}). \tag{1}$$

This is why we cannot construct the quantum algebra $U_q(\mathrm{so}_{n,1})$ in the frame of these approaches and cannot construct Gel'fand–Tsetlin bases in the representation spaces. In order to obtain inclusion (1) it was proposed in [1] another q-deformation of the classical universal enveloping algebra $U(\mathrm{so}(n, \mathbf{C}))$. The classical algebra $U(\mathrm{so}(n, \mathbf{C}))$ is generated by the elements $I_{i,i-1}$, $i = 2, 3, \cdots, n$, that satisfy the relations

$$I_{i,i-1}I_{i+1,i}^2 - 2I_{i+1,i}I_{i,i-1}I_{i+1,i} + I_{i+1,i}^2 I_{i,i-1} = -I_{i,i-1}, \tag{2}$$

$$I_{i,i-1}^2 I_{i+1,i} - 2I_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I_{i,i-1}^2 = -I_{i+1,i}, \tag{3}$$

$$[I_{i,i-1}, I_{j,j-1}] = 0, \quad |i - j| > 1. \tag{4}$$

They follow from the well-known commutation relations for the generators I_{ij} of the Lie algebra $so(n, \mathbf{C})$ (see the paper by Gel'fand and Tsetlin [10]).

The approach of the paper [1] to the q-deformed orthogonal algebra consists in a q-deformation of the associative algebra $U(so(n, \mathbf{C}))$ by deforming relations (2)-(4). The q-deformed relations are of the form

$$I_{i,i-1}I_{i+1,i}^2 - aI_{i+1,i}I_{i,i-1}I_{i+1,i} + I_{i+1,i}^2 I_{i,i-1} = -I_{i,i-1}, \tag{5}$$

$$I_{i,i-1}^2 I_{i+1,i} - aI_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I_{i,i-1}^2 = -I_{i+1,i}, \tag{6}$$

$$[I_{i,i-1}, I_{j,j-1}] = 0, \quad |i - j| > 1, \tag{7}$$

where $a = q + q^{-1} = (q^2 - q^{-2})/(q - q^{-1})$ and $[\cdot, \cdot]$ denotes the usual commutator. Obviously, in the limit $q \to 1$ formulas (5)–(7) give relations (2)–(4). Remark that relations (5) and (6) differ from the q-deformed Serre relations in the approach of Jimbo and Drinfeld to quantum orthogonal algebras by appearance of nonzero right hand side and by a possibility of reduction (1). Below, by the algebra $U_q'(so_n)$ we mean the q-deformed algebra defined by formulas (5)–(7). Unfortunately, the algebra $U_q'(so_n)$ does not have a Hopf algebra structure. But it can be embedded into the Hopf algebra $U_q(sl_n)$ and is a Hopf ideal in $U_q(sl_n)$ [3].

As in the classical case, the q-algebras $U_q'(so_3)$ and $U_q'(so_4)$ can also be described in terms of bilinear relations (q-commutators). In fact, defining the algebra $U_q'(so_3)$ by relations (5)–(7) we have only two generators I_{21} and I_{32}. However, we can define the third element I_{31} according to the formula

$$I_{31} = q^{1/2}I_{21}I_{32} - q^{-1/2}I_{32}I_{21} \tag{8}$$

(see [2]). Then by the algebra $U_q'(so_3)$ we mean the associative algebra generated by the elements I_{21}, I_{32} and I_{31} which satisfy the relations

$$q^{1/2}I_{21}I_{32} - q^{-1/2}I_{32}I_{21} = I_{31}, \tag{9}$$

$$q^{1/2}I_{31}I_{21} - q^{-1/2}I_{21}I_{31} = I_{32}, \tag{10}$$

$$q^{1/2}I_{32}I_{31} - q^{-1/2}I_{31}I_{32} = I_{21}. \tag{11}$$

It is clear that if the generators I_{21}, I_{32} and I_{31} satisfy the relations (9)–(11), then the pair I_{21} and I_{32} satisfies the trilinear relations (5) and (6). Remark that the algebra given by relations (9)–(11) coincides with the cyclically symmetric algebra defined in [11].

The q-deformed algebra $U_q'(so_4)$ is generated by the generators I_{21}, I_{32} and I_{43}. Moreover, for the first two generators everything, said above $U_q'(so_3)$, is true. Thus, the inclusion $U_q'(so_3) \subset U_q'(so_4)$ takes place. The generators I_{21} and I_{43} mutually commute (see relation (7)) and the pair I_{32}, I_{43} in turn must satisfy relations (5) and (6). Again, $U_q'(so_4)$ can be also given in terms of bilinear q-commutators. Namely, we can add to the triple of generators I_{21}, I_{32} and I_{43} the element I_{31} from (8) and the elements I_{42}, I_{41} defined as

$$I_{42} = q^{1/2}I_{32}I_{43} - q^{-1/2}I_{43}I_{32},$$

$$I_{41} = q^{1/2}I_{31}I_{43} - q^{-1/2}I_{43}I_{31} = q^{1/2}I_{21}I_{42} - q^{-1/2}I_{42}I_{21}.$$

Various real forms of the algebra $U_q'(so_4)$ are obtained by introducing corresponding $*$-structures (antilinear antihomomorphisms). The real form $U_q'(so_{2,2})$ is defined by the $*$-structure

$$I_{21}^* = -I_{21}, \quad I_{32}^* = I_{32}, \quad I_{34}^* = -I_{34}.$$

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The relations $I_{21}^* = -I_{21}$, $I_{32}^* = -I_{32}$, $I_{34}^* = I_{34}$ determine the real form $U_q'(\mathrm{so}_{3,1})$.

Everywhere below we assume that q is a positive real number.

3. REPRESENTATIONS OF THE ALGEBRA $U_q'(\mathrm{so}_4)$

Let us describe irreducible finite dimensional representations of $U_q'(\mathrm{so}_4)$ when q is not a root of unity. They are described by using representations of $U_q'(\mathrm{so}_3)$.

Irreducible finite dimensional representations of $U_q'(\mathrm{so}_3)$ are given by integral or half-integral nonnegative number l. We denote these representations by T_l. The carrier space of the representation T_l has the orthonormal basis $\{|m\rangle, \ m = l, l-1, \cdots, -l\}$, and the operators $T_l(I_{21})$ and $T_l(I_{32})$ act upon this basis as

$$T_l(I_{21})|m\rangle = \mathrm{i}[m]|m\rangle, \tag{12}$$

$$T_l(I_{32})|m\rangle = d(m)([l-m][l+m+1])^{1/2}|m+1\rangle - d(m-1)([l-m+1][l+m])^{1/2}|m-1\rangle, \tag{13}$$

where

$$d(m) = ([m][m+1]/[2m][2m+2])^{1/2}$$

and $[a]$ denotes a q-number defined by

$$[a] = (q^a - q^{-a})/(q - q^{-1}).$$

If q is positive then these representations exhaust all irreducible finite dimensional representations of $U_q'(\mathrm{so}_3)$. For other values of q there exist irreducible finite dimensional representations which are not equivalent to these representations (see, for example, [4]) but we shall not need them.

As in the case of the Lie group $SO(4)$, finite dimensional irreducible representations T_{rs} of the q-deformed algebra $U_q'(\mathrm{so}_4)$ are given by two integral or half-integral numbers r and s such that $r \geq |s| \geq 0$ (see [6]). Restriction of T_{rs} onto the subalgebra $U_q'(\mathrm{so}_3)$ decomposes into the sum of the irreducible representations T_l of this subalgebra for which $l = |s|, |s|+1, \cdots, r$. Uniting the bases of the subspaces of the irreducible representations T_l of $U_q'(\mathrm{so}_3)$ we obtain the basis of the carrier space V_{rs} of the representation T_{rs} of $U_q'(\mathrm{so}_4)$. Thus, the corresponding orthonormal basis of V_{rs} consists of the vectors

$$|l, m\rangle, \quad |s| \leq l \leq r, \quad m = -l, -l+1, \cdots, l.$$

The operator $T_{rs}(I_{43})$ acts upon these vectors by the formula

$$T_{rs}(I_{43})|l, m\rangle = \mathrm{i}\frac{[r+1][s][m]}{[l][l+1]}|l, m\rangle$$

$$+ \left(\frac{[r-l][l+s+1][l-s+1][l+m+1][l-m+1]}{[r+l+2]^{-1}[l+1]^2[2l+1][2l+3]}\right)^{1/2}|l+1, m\rangle$$

$$- \left(\frac{[r+l+1][l+s][l-s][l+m][l-m]}{[r-l+1][l]^2[2l-1][2l+1]}\right)^{1/2}|l-1, m\rangle, \tag{14}$$

where numbers in the square brackets are q-numbers. The operators $T_{rs}(I_{21})$ and $T_{rs}(I_{32})$ act upon the basis vectors by formulas (12) and (13). Formulas (12), (13) and (14) completely determine the representation T_{rs}.

4. DIAGONALIZATION OF THE OPERATOR $T_{rs}(I_{43})$

We shall need the representations T_{rs} in another form. To obtain it we diagonalize the operator $T_{rs}(I_{43})$. In the next section this result is used for obtaining representations T_{rs} in the bases corresponding to restriction upon the subalgebra $U_q'(\mathrm{so}_2) + U_q'(\mathrm{so}_2)$. It is more convenient to

deal with the selfadjoint operator $L = -iT_{rs}(I_{43})$, $i = \sqrt{-1}$. Replacing the vectors $|l,m\rangle$ by $|l,m\rangle' = i^{-l}|l,m\rangle$ we obtain that L acts upon the vectors $|l,m\rangle'$ by formula (14) in which the sign $-$ of the third summand is replaced by $+$ and the first summand is multipled by $-i$.

The space V_{rs} can be decomposed into the sum $V_{rs} = \sum_{m=-s}^{r} \oplus\, V_m$, where V_m is spanned by the vectors $|l,m\rangle$ with fixed m. Let us find the spectrum and the eigenvectors

$$|x,m\rangle' = \sum_{l=k}^{r} P_{l-k}(x)|l,m\rangle, \quad k = \max\,(|m|,\ |s|) \tag{15}$$

of the operator L on the subspace V_m:

$$L|x,m\rangle' = [x]|x,m\rangle', \tag{16}$$

where $[x]$ is a q-number. Formula (14) is symmetric with respect to permutation of s and m and to change of signs at m and s. Therefore, we may assume, without loss of generality, that s and m are positive and that $s \geq m$.

Substituting expression (15) for $|x,m\rangle'$ into (16) and acting by L upon $|l,m\rangle$ we easily find that vector (15) is an eigenvector of L with the eigenvalue $[x]$ if P_{l-k} satisfy the recurrence relation

$$\left(\frac{[u][n+2s+1][n+1][n+s+m+1][n+s-m+1]}{[r+n+s+2]^{-1}[n+s+1]^2[2n+2s+1][2n+2s+3]}\right)^{1/2} P_{n+1}(x)$$

$$+\left(\frac{[r+n+s+1][r-n-s+1][n+2s][n][n+s+m]}{[n+s-m]^{-1}[n+s]^2[2n+2s-1][2n+2s+1]}\right)^{1/2} P_{n-1}(x)$$

$$+\frac{[r+1][s][m]}{[n+s][n+s+1]}P_n(x) = [x]P_n(x) \tag{17}$$

(here $u = r - n - s$, $n = l - k$) and the initial conditions $P_0(x) = 1$, $P_{-1}(x) = 0$.

Making in (17) the substitution

$$P_n(x) = -q^c\left(\frac{[n+2s]![n+s+m]![2n+2s+1]}{[n]![n+s-m]![r-n-s]![r+n+s+1]!}\right)^{1/2} P'_n(x)$$

where $c = s + m - r$, we reduce (17) to the recurrence relation

$$\frac{(1-Q^{n+2s+1})(1-Q^{n+s+m+1})(1-Q^{n-r+s})(1+Q^{n+s+1})}{(1-Q^{2n+2s+1})(1-Q^{2n+2s+2})}P'_{n+1}(x)$$

$$-\frac{Q^{s+m-r}(1-Q^n)(1-Q^{r+n+s+1})(1+Q^{n+s})(1-Q^{n+s-m})}{(1-Q^{2n+2s+1})(1-Q^{2n+2s})}P'_{n-1}(x)$$

$$-\frac{Q^{n-r+s}(1-Q^{r+1})(1-Q^m)(1-Q^s)}{(1-Q^{n+s})(1-Q^{n+s+1})}P'_n(x) = \frac{q-q^{-1}}{Q^{(r-s-m)/2}}[x]P'_n(x),$$

where $Q := q^2$. Comparing this formula with recurrence relation (7.5.2) from the book of Gasper and Rahman [13] for q-Racah polynomials

$$R_n(\mu(y);\ \alpha,\beta,\gamma,\delta|Q) = {}_4\varphi_3\left(\begin{array}{c}Q^{-y},Q^{y+1}\gamma\delta,Q^{-n},Q^{n+1}\alpha\beta\\ \alpha Q,\quad \beta\delta Q,\quad \gamma Q\end{array};\ Q,Q\right)$$

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(here $_4\varphi_3$ is a basic hypergeometric function which can be found in [13]) at

$$\alpha = \beta = -Q^s, \ \gamma = Q^{s+m}, \ \delta = -Q^{-r-1}, \tag{18}$$

after cumbersome transformations we conclude that

$$P'_n(x) = R_n(\mu(y); \ \alpha, \beta, \gamma, \delta | Q),$$

where α, β, γ, δ are given by formulas (18) and $x = (r - s - m) - 2y$. Thus, the polynomials $P_n(x)$ from (17) normalized by the condition $P_0(x) = 1$ are of the form

$$P_n(x) = N^{1/2} R_n(\mu(y); \ -Q^s, -Q^s, Q^{s+m}, -Q^{-r-1} | Q), \tag{19}$$

$$N = \frac{[n + 2s]![n + s + m]![2n + 2s + 1][s - m]![r - s]![r + s + 1]!}{[n]![n + s - m]![r - n - s]![r + n + s + 1]![2s]![s + m]![2s + 1]},$$

where $x = (r - s - m) - 2y$. The variable y takes the values $0, 1, 2, \cdots, r - s$. Therefore, the spectrum of L on the subspace V_m consists of the points

$$[r - s - m], \ [r - s - 2 - m], \ [r - s - 4 - m], \ \cdots, \ [-(r - s) - m]. \tag{20}$$

The corresponding eigenvectors are determined by formulas (15) and (19). The orthogonality relation for the polynomials $P_n(x)$ follows from the orthogonality of q-Racah polynomials (see [13]) and is of the form

$$\sum_{y=0}^{r-s} P_n(x) P_k(x) W(x) = \delta_{nk}. \tag{21}$$

Here $W(x)$ is equal to the expression

$$\frac{[4y + 2k - 2r][2y + 2k - 2r - 2]!![2y + 2s]!![2r - 2y]!![r - m - y]!}{[2y + 2k - 2r][y + k - r - 1]![y + s]![2y + 2m]!![r - y]![r - s - y]![y]!}$$

$$\times [y + m]![k + y]![2s + 1]!!([s]!)^2 [r - s]([2s]!![s - m]![k]![r + s + 1]!)^{-1},$$

where $k = s + m$, $[n]! = [n][n - 1] \cdots [1]$ and $[n]!! = [n][n - 2][n - 4] \cdots [1]$ or $[2]$.

Formula (21) shows that vectors (15) are not normalized. The vectors $|x, m\rangle = W(x)^{1/2}|x, m\rangle'$ are normal and due to formula (16) we have

$$T_{rs}(I_{43})|x, m\rangle = \mathrm{i}[x]|x, m\rangle. \tag{22}$$

Joining spectra (20) for all subspaces V_m, we obtain the spectrum of the operator T_{rs}, and therefore the spectrum of the operator $T_{rs}(I_{43})$.

5. Representations T_{rs} in the basis $|x, m\rangle$

The operator $T_{rs}(I_{43})$ acts upon the basis vectors $|x, m\rangle$ by formula (22). It is clear from formulas (13) and (15) that

$$T_{rs}(I_{21})|x, m\rangle = \mathrm{i}[m]|x, m\rangle. \tag{23}$$

Thus, to have the representation T_{rs} in the basis $|x, m\rangle$, we must find the action formula for the operator $T_{rs}(I_{32})$ upon this basis.

Since

$$|x, m\rangle = \sum_{l=s}^{r} P_{l-s}^m(x)|l, m\rangle, \tag{24}$$

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with $P^m_{l-s}(x) = W(x)^{1/2}P_{l-s}(x)$, then due to formula (13) we have

$$T_{rs}(I_{32})|x,m\rangle = d(m)\sum_{l=s}^{r}P^m_{l-s}(x)([l-m][l+m+1])^{1/2}|l,m+1\rangle$$

$$-d(m-1)\sum_{l=s}^{r}P^m_{l-s}(x)([l-m+1][l+m])^{1/2}|l,m-1\rangle. \qquad (25)$$

Applying to $([l-m][l+m+1])^{1/2}P^m_{l-s}(x)$ recurrence relation (7.2.14) of [13] with

$$a = Q^{m-r-1}, \quad b = -Q^s, \quad c = d = -Q^m, \quad n = (r-s-m-x)/2, \quad j = l-m$$

and using the equalities

$$[2x]/[x] = q^x + q^{-x}, \qquad (q^{a+b} \pm q^{-(a+b)})(q^{a-b} \mp q^{-(a-b)}) = [2a] \mp [2b],$$

after some calculations we obtain for the first summand of the right hand side of (25) the expression

$$d(m)d(x-1)\{([r+1]+[s-m+x-1])([r+1]+[s+m-x+1])\}^{1/2}\sum_{l=s}^{r}P^{m+1}_{l-s}(x-1)|l,m+1\rangle$$

$$-d(m)d(x)\{([r+1]-[s+m+x+1])([r+1]-[s-m-x-1])\}^{1/2}\sum_{l=s}^{r}P^{m+1}_{l-s}(x+1)|l,m+1\rangle. \qquad (26)$$

To transform the second summand on the right hand side of (25) we apply to the basic hypergeometric function $_4\varphi_3$ from the expression for $P^m_{l-s}(x)$ the transformation

$$_4\varphi_3\left(\begin{matrix}Q^{-N},\alpha,\beta,\gamma\\ \delta,\ \sigma,\ \rho\end{matrix};Q,Q\right) = \frac{(\sigma/\alpha;Q)_N(\rho/\alpha;Q)_N}{(\sigma;Q)_N(\rho;Q)_N\alpha^{-N}}\,{}_4\varphi_3\left(\begin{matrix}Q^{-N},\alpha,\ \delta/\beta,\ \delta/\gamma\\ \delta,\alpha Q^{1-N}/\sigma,\alpha Q^{1-N}/\rho\end{matrix};Q,Q\right)$$

(see [13]), where $N = l - s$ and

$$\alpha = Q^{l+s+1}, \quad \beta = -Q^{(s-r+m-x)/2}, \quad \gamma = Q^{(s-r+m+x)/2},$$

$$\delta = Q^{s-r}, \quad \sigma = -Q^{s+1}, \quad \rho = Q^{s+m+1}.$$

Here $(a;Q)_n = (1-Q)(1-aQ)(1-aQ^2)\cdots(1-aQ^{n-1})$. Now we apply to $([l-m+1][l+m])^{1/2}P^m_{l-s}(x)$ the same recurence relation (7.2.14) of [13] with

$$a = Q^{-(r+m+1)}, \quad b = -Q^{-s}, \quad c = d = -Q^m, \quad n = (r-s+m+x)/2, \quad j = l+m.$$

Then the second summand of the right hand side of (25) takes the form

$$d(m-1)d(x)\{([r+1]+[s+m-x-1])([r+1]+[s-m+x+1])\}^{\frac{1}{2}}\sum_{l=s}^{r}P^{m-1}_{l-s}(x+1)|l,m-1\rangle$$

$$-d(m-1)d(x-1)\{([r+1]-[s-m-x+1])([r+1]-[s+m+x-1])\}^{\frac{1}{2}}\sum_{l=s}^{r}P^{m-1}_{l-s}(x-1)|l,m-1\rangle \qquad (27)$$

We substitute expressions (26) and (27) into (25) and take into account formula (24). As a result, we find that the operator $T_{rs}(I_{32})$ acts upon the vectors $|x,m\rangle$ as

$$T_{rs}(I_{32})|x,m\rangle =$$

$$= d(m)d(x-1)\{([r+1]+[s-m+x-1])([r+1]+[s+m-x+1])\}^{1/2}|x-1,m+1\rangle$$

$$-d(m)d(x)\{([r+1]-[s+m+x+1])([r+1]-[s-m-x-1])\}^{1/2}|x+1,m+1\rangle$$

$$+d(m-1)d(x-1)\{([r+1]-[s-m-x+1])([r+1]-[s+m+x-1])\}^{1/2}|x-1,m-1\rangle$$

$$-d(m-1)d(x)\{([r+1]+[s+m-x-1])([r+1]+[s-m+x+1])\}^{1/2}|x+1,m-1\rangle. \qquad (28)$$

Now we completely determined representations T_{rs} of $U_q'(\text{so}_4)$ with respect to the basis corresponding to reduction onto the subalgebra $U_q'(\text{so}_2) + U_q'(\text{so}_2)$.

6. Infinite dimensional representations of $U_q'(\text{so}_{2,2})$

Let us first define infinite dimensional linear representations of $U_q'(\text{so}_{2,2})$. By a linear representation T of the algebra $U_q'(\text{so}_{2,2})$ we mean a homomorphism of $U_q'(\text{so}_{2,2})$ into the algebra of linear operators (bounded or unbounded) on a Hilbert space H, defined on an everywhere dense invariant subspace D, such that

(a) the operators $T(I_{21})$ and $T(I_{43})$ can be simulteniously diagonalized,
(b) eigenvalues of $T(I_{21})$ and $T(I_{43})$ have finite multiplicities,
(c) eigenvectors of $T(I_{21})$ and $T(I_{43})$ belong to D.

A representation T of $U_q'(\text{so}_{2,2})$ is called a $*$-representation if the operators $T(I_{21})$, $T(I_{32})$ and $T(I_{43})$ satisfy on D the relations

$$T(I_{21})^* = -T(I_{21}), \quad T(I_{32})^* = T(I_{32}), \quad T(I_{43})^* = -T(I_{43}).$$

As in the case of representations of compact and noncompact real Lie groups, by making use of analytical continuation in parameters giving representations we can obtain infinite dimensional representations of the q-deformed algebra $U_q'(\text{so}_{2,2})$ from the representations T_{rs} of $U_q'(\text{so}_4)$. In this way, we obtain the representations $T_{\sigma\tau}^\epsilon$, $\sigma \in \mathbf{C}$, $\tau \in \mathbf{C}$, $\epsilon \in \{0, 1\}$, of $U_q'(\text{so}_{2,2})$ which act on the Hilbert spaces H_ϵ with the orthonormal basis

$$|x,m\rangle, \quad x \in \frac{1}{2}\mathbf{Z}, \quad m \in \frac{1}{2}\mathbf{Z}, \quad x+m \equiv \epsilon \pmod{2}.$$

The operators $T_{\sigma\tau}^\epsilon(I_{21})$ and $T_{\sigma\tau}^\epsilon(I_{43})$ act upon these basis vectors by formulas (22) and (23). For the operator $T_{\sigma\tau}^\epsilon(I_{32})$ we have

$$T_{\sigma\tau}^\epsilon(I_{32})|x,m\rangle$$

$$= d(m)d(x-1)\{([\sigma+1]+[\tau-m+x-1])([\sigma+1]+[\tau+m-x+1])\}^{\frac{1}{2}}|x-1,m+1\rangle$$

$$-d(m)d(x)\{([\sigma+1]-[\tau+m+x+1])([\sigma+1]-[\tau-m-x-1])\}^{\frac{1}{2}}|x+1,m+1\rangle$$

$$+d(m-1)d(x-1)\{([\sigma+1]-[\tau-m-x+1])([\sigma+1]-[\tau+m+x-1])\}^{\frac{1}{2}}|x-1,m-1\rangle$$

$$-d(m-1)d(x)\{([\sigma+1]+[\tau+m-x-1])([\sigma+1]+[\tau-m+x+1])\}^{\frac{1}{2}}|x+1,m-1\rangle.$$

It is more convenient to write down the last formula in the form

$$T_{\sigma\tau}^\epsilon(I_{32})|x,m\rangle$$

$$= \left(\frac{[\sigma-\tau+m-x+2][\sigma-\tau-m+x][(\sigma+\tau+m-x+2)/2]}{[(\sigma-\tau+m-x+2)/2][(\sigma-\tau-m+x)/2][(\sigma+\tau-m+x)/2]^{-1}} \right)^{1/2}$$

$$\times d(m)d(x-1)|x-1,m+1\rangle$$

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$$-\left(\frac{[\sigma+\tau+m+x+2][\sigma+\tau-m-x][(\sigma-\tau+m+x+2)/2]}{[(\sigma+\tau+m+x+2)/2][(\sigma+\tau-m-x)/2][(\sigma-\tau-m-x)/2]^{-1}}\right)^{1/2}$$
$$\times d(m)d(x)|x+1,m+1\rangle$$

$$+\left(\frac{[\sigma+\tau-m-x+2][\sigma+\tau+m+x][(\sigma-\tau-m-x+2)/2]}{[(\sigma+\tau-m-x+2)/2][(\sigma+\tau+m+x)/2][(\sigma-\tau+m+x)/2]^{-1}}\right)^{1/2}$$
$$\times d(m-1)d(x-1)|x-1,m-1\rangle$$

$$-\left(\frac{[\sigma-\tau-m+x+2][\sigma-\tau+m-x][(\sigma+\tau-m+x+2)/2]}{[(\sigma-\tau-m+x+2)/2][(\sigma-\tau+m-x)/2][(\sigma+\tau+m-x)/2]^{-1}}\right)^{1/2}$$
$$\times d(m-1)d(x)|x+1,m-1\rangle. \tag{29}$$

In every summand here there are two expressions of the form $[\sigma-\tau-m+x]/[(\sigma-\tau-m+x)/2]$. This expression is equal to $q^{(\sigma-\tau-m+x)/4}+q^{-(\sigma-\tau-m+x)/4}$.

Introducing the notation

$$\gamma=\sigma+\tau+2,\quad \delta=\sigma-\tau+2,\quad M=x+m,\quad N=m-x,$$

where M and N are even if $\epsilon=0$ and odd if $\epsilon=1$, and denoting the basis elements $|x,m\rangle$ by $|M,N\rangle'$, we obtain after multiplication of $|M,N\rangle'$ by the appropriate factors that

$$T^\epsilon_{\sigma\tau}(I_{32})|M,N\rangle = ([(M+N)/2][(M-N)/2])/([M+N][M-N])$$

$$\times\{(q^{(\delta+N)/2}+q^{-(\delta+N)/2})[(\gamma+N)/2]|M,N+2\rangle$$
$$-(q^{(\gamma+M)/2}+q^{-(\gamma+M)/2})[(\delta+M)/2]|M+2,N\rangle$$
$$+(q^{(\gamma-M)/2}+q^{-(\gamma-M)/2})[(\delta-M)/2]|M-2,N\rangle$$
$$-(q^{(\delta-N)/2}+q^{-(\delta-N)/2})[(\gamma-N)/2]|M,N-2\rangle\},$$

where $|M,N\rangle$ are the basis elements $|M,N\rangle'$ with the appropriate factors.

Setting
$$k=M/2,\quad l=N/2,\quad c=\gamma/2,\quad b=\delta/2$$

and denoting the basis elements $|M,N\rangle$ by $|k,l\rangle$ and the representations $T^\epsilon_{\sigma\tau}$ by T^ϵ_{bc} we obtain the representations in the form

$$T^\epsilon_{bc}(I_{32})|k,l\rangle = ([k+l][k-l])/([2(k+l)][2(k-l)])$$

$$\times\{(q^{l+b}+q^{-l-b})[l+c]|k,l+1\rangle - (q^{k+c}+q^{-k-c})[k+b]|k+1,l\rangle$$
$$+(q^{-k+c}+q^{k-c})[b-k]|k-1,l\rangle - (q^{b-l}+q^{l-b})[c-l]|k,l-1\rangle\}, \tag{30}$$

$$T^\epsilon_{bc}(I_{21})|k,l\rangle = \sqrt{-1}[k+l]|k,l\rangle,\qquad T^\epsilon_{bc}(I_{43})|k,l\rangle = \sqrt{-1}[k-l]|k,l\rangle. \tag{31}$$

Note that the basis consists of the vectors $|k,l\rangle$, where k and l are integral if $\epsilon=0$ and half-integral (half of an odd integer) if $\epsilon=1$. We consider that the invariant everywhere dense subspace D_ϵ of H_ϵ coincides with the span of all vectors $|k,l\rangle$ from H_ϵ.

Remark that the operators $T^\epsilon_{bc}(I_{21})$ and $T^\epsilon_{bc}(I_{43})$ are unbounded and the operator $T^\epsilon_{bc}(I_{32})$ is bounded.

Our aim is to study representations T^ϵ_{bc} of $U'_q(\mathrm{so}_{2,2})$. We say that T^ϵ_{bc} is irreducible if it is algebraically irreducible on the subspace D_ϵ.

Theorem 1. *The representation T^ϵ_{bc} is irreducible if and only if no of the numbers b and c coincides with any of the numbers n, $n + \mathrm{i}\pi r/2h$, $n, r \in \mathbf{Z}$, where h is defined by $q = \exp h$.*

Proof is given as in the case of representations of semisimple Lie algebras (see, for example, [14], Chapter 7).

There exist equivalence relations in the set of representations T^ϵ_{bc}. One type of equivalences appears because of the periodicity of the function $w(z) = [z]$, where $[z] = (q^z - q^{-z})/(q - q^{-1})$. If $q = \exp h$, then the function $w(z)$ is periodic with period $2\pi\mathrm{i}/h$. Therefore, it follows from (30) and (31) that

$$T^\epsilon_{bc} = T^\epsilon_{b+2\pi\mathrm{i}/h,c} = T^\epsilon_{b,c+2\pi\mathrm{i}/h}. \tag{32}$$

For the function $w(z)$ we also have $w(z) = -w(z + \pi\mathrm{i}/h)$. For this reason, replacement of b (respectively, of c) by $b + \pi\mathrm{i}/h$ (respectively, by $c + \pi\mathrm{i}/h$) in the relation (30) changes only signs near vectors, and the representations T^ϵ_{bc} and $T^\epsilon_{b+\pi\mathrm{i}/h,c}$ (respectively, $T^\epsilon_{b,c+\pi\mathrm{i}/h}$) are equivalent and the equivalence operator is diagonal with respect to the basis $\{|k,l\rangle\}$ with numbers ± 1 on the main diagonal. Thus,

$$T^\epsilon_{bc} \sim T^\epsilon_{b+\pi\mathrm{i}/h,c} \sim T^\epsilon_{b,c+\pi\mathrm{i}/h}. \tag{33}$$

If the representation T^ϵ_{bc} is irreducible, then we also have the equivalences

$$T^\epsilon_{bc} \sim T^\epsilon_{b,-c+1} \sim T^\epsilon_{-b+1,c}. \tag{34}$$

The equivalence operators are diagonal in the basis $\{|k,l\rangle\}$ and their matrix elements are calculated in the same way as in the case of representations of Lie algebras (see, for example, [15], Sect. 6.4.4).

Using the method of [15], Sect. 6.4, it is easy to prove that any equivalence relation between irreducible representations in the set of representations T^ϵ_{bc} is a composition of equivalence relations given above.

Taking into account the equivalences (32)–(34), everywhere below we assume (without loosing the generality) that $0 \le \operatorname{Im} b < \pi\mathrm{i}/h$, $0 \le \operatorname{Im} c < \pi\mathrm{i}/h$, $\operatorname{Re} b \ge 1/2$ and $\operatorname{Re} c \ge 1/2$.

7. Irreducible subrepresentations of T^ϵ_{bc}

In order to find irreducible constituents of reducible representations T^ϵ_{bc} we reduce these representations to the form (29). If the representation T^ϵ_{bc} is irreducible, then for the operator $T^\epsilon_{bc}(I_{32})$ we have

$$T^\epsilon_{bc}(I_{32})|k,l\rangle' = ([k+l][k-l])/([2(k+l)][2(k-l)])$$

$$\times\{((q^{l+b} + q^{-l-b})(q^{b-l-1} + q^{l-b+1})[l+c][l-c+1])^{1/2}|k,l+1\rangle'$$

$$-((q^{k+c} + q^{-k-c})(q^{c-k-1} + q^{k-c+1})[k+b][k-b+1])^{1/2}|k+1,l\rangle'$$

$$-((q^{k+c-1} + q^{-k-c+1})(q^{c-k} + q^{k-c})[k-b][k+b-1])^{1/2}|k-1,l\rangle'$$

$$+((q^{l+b-1} + q^{-l-b+1})(q^{b-l} + q^{l-b})[l-c][l+c-1])^{1/2}|k,l-1\rangle'\}, \tag{35}$$

where the vectors $|k,l\rangle'$ are obtained from $|k,l\rangle$ by multiplication by appropriate factors. The operators $T^\epsilon_{bc}(I_{21})$ and $T^\epsilon_{bc}(I_{43})$ are given in the basis $\{|k,l\rangle'\}$ by the same formulas (31).

We analytically continue formula (35) to the velues of b and c for which the operators $T^\epsilon_{bc}(I_{32})$ give reducible representations (that is, to those values of b and c which are excluded in Theorem 1). As a result, we obtain the new operators $\tilde{T}^\epsilon_{bc}(I_{32})$ which gives with the operators $T^\epsilon_{bc}(I_{21})$ and $T^\epsilon_{bc}(I_{43})$ an irreducible representation $\tilde{T}^\epsilon_{bc}$ of $U'_q(\mathrm{so}_{2,2})$. As in the similar situation of the case of representations of Lie algebras (see [14]), the reducible representations $\tilde{T}^\epsilon_{bc}$ and T^ϵ_{bc} consist of

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the same irreducible components. It is more convenient to look for irreducible components of the reducible representations $\tilde{T}^{\epsilon}_{bc}$. Depending on values of b and c we differ 5 cases.

Case 1: Let b be integral if $\epsilon = 0$ and half-integral if $\epsilon = 1$, and let $c \neq c', c' + i\pi/2h$, where $c' \in \mathbf{Z}$ if $\epsilon = 0$ and c' is half-integral if $\epsilon = 1$. (Recall that we assumed that $b \geq 1/2$ and $0 \leq \text{Im}\, c < \pi i/h$.) Then the second and third summands on the right hand side of (35) vanish for $k = -b$, $k = b - 1$ and $k = b$, $k = -b + 1$, respectively. By the same reasoning as in the case of representations of Lie algebras (see, for example, [14], Chap. 7) we find that these vanishings of summands lead to appearence of the irreducible invariant subspaces H^0_{bc}, H^+_{bc} and H^-_{bc} of the representation space H_{ϵ} spanned by all the basis vectors $|k, l\rangle'$ with $-b < k < b$, $k \geq b$ and $k \leq -b$, respectively. (Note that if $b = 1/2$, then the subspace H^0_{bc} is empty.) We denote the corresponding irreducible subrepresentations of $\tilde{T}^{\epsilon}_{bc}$ by D^0_{bc}, D^+_{bc} and D^-_{bc}, respectively. We have

$$\tilde{T}^{\epsilon}_{bc} \sim D^0_{bc} \oplus D^+_{bc} \oplus D^-_{bc} \quad \text{if} \quad b > 1/2 \quad \text{and} \quad \tilde{T}^{\epsilon}_{1/2,c} \sim D^+_{1/2,c} \oplus D^-_{1/2,c}.$$

Note that if $b = 1$ then the subspace H^0_{bc} is spanned by the basis vectors $|0, l\rangle' \equiv |l\rangle$. In this case the operator $D^0_{1,c}(I_{32})$ is given by the formula

$$D^0_{1,c}(I_{32})|l\rangle = ([l]^2/[2l]^2)\{((q^{l+1} + q^{-l-1})(q^{-l} + q^l)[l + c][l - c + 1])^{1/2}|l + 1\rangle$$

$$+ ((q^l + q^{-l})(q^{-l+1} + q^{l-1})[l - c][l + c - 1])^{1/2}|l - 1\rangle\}. \tag{36}$$

Case 2: Let c be integral if $\epsilon = 0$ and half-integral if $\epsilon = 1$, and let $b \neq b', b' + i\pi/2h$, where $b' \in \mathbf{Z}$ if $\epsilon = 0$ and b' is half-integral if $\epsilon = 1$. In this case the representation space H_{ϵ} has the irreducible invariant subspaces H^0_{bc}, H^+_{bc} and H^-_{bc} spanned by the basis vectors $|k, l\rangle'$ with $-c < l < c$, $l \geq c$ and $l \leq -c$, respectively. We denote the corresponding irreducible subrepresentations of $\tilde{T}^{\epsilon}_{bc}$ by F^0_{bc}, F^+_{bc} and F^-_{bc}, respectively. We have

$$\tilde{T}^{\epsilon}_{bc} \sim F^0_{bc} \oplus F^+_{bc} \oplus F^-_{bc} \quad \text{if} \quad b > 1/2 \quad \text{and} \quad \tilde{T}^{\epsilon}_{b,1/2} \sim F^+_{b,1/2} \oplus F^-_{b,1/2}.$$

The representation $F^0_{b,1}$ acts on the space spanned by the vectors $|k, 0\rangle \equiv |k\rangle$ and the operator $F^0_{b,1}(I_{32})$ is given by the formula similar to the relation (36).

Case 3: Let b be as in Case 1 and let c be of the form $c = c' + i\pi/2h$, where $c' \in \mathbf{Z}$ if $\epsilon = 0$ and c' is half-integral if $\epsilon = 1$ and such that $c' > b$. Then the second summand on the right hand side of (35) vanishes when $k = -b$, $k = b - 1$, $k = -c$, $k = c - 1$ and the third summand vanishes when $k = b$, $k = -b + 1$, $k = c$, $k = -c + 1$. This leads to appearence of five irreducible invariant subspaces H^0_{bc}, H^+_{bc}, H^{++}_{bc}, H^-_{bc}, H^{--}_{bc} in H_{ϵ} spanned by all the basis vectors $|k, l\rangle'$ with $-b < k < b$, $b \leq k < c'$, $k \geq c'$, $-c' < k \leq -b$ and $k \leq -c'$, respectively. (Note that if $b = 1/2$, then subspace H^0_{bc} is empty.) We denote the corresponding irreducible subrepresentations of $\tilde{T}^{\epsilon}_{bc}$ by Q^0_{bc}, Q^+_{bc}, Q^{++}_{bc}, Q^-_{bc}, Q^{--}_{bc}. We have

$$\tilde{T}^{\epsilon}_{bc} \sim Q^0_{bc} \oplus Q^+_{bc} \oplus Q^{++}_{bc} \oplus Q^-_{bc} \oplus Q^{--}_{bc} \quad \text{if} \quad b > 1/2,$$

$$\tilde{T}^{\epsilon}_{1/2,c} \sim Q^+_{1/2,c} \oplus Q^{++}_{1/2,c} \oplus Q^-_{1/2,c} \oplus Q^{--}_{1/2,c}.$$

If $b = 1$, then the subspace H^0_{bc} is spanned by the vectors $|0, l\rangle' \equiv |l\rangle$ and the operator $Q^0_{bc}(I_{32})$ is given by the formula similar to the relation (36). If $c' = b + 1$, then the subspace H^+_{bc} is spanned by the basis vectors $|b, l\rangle' \equiv |l\rangle$ and the operator $Q^+_{bc}(I_{32})$ is given by

$$Q^+_{bc}(I_{32})|l\rangle = -([b + l][b - l])/([2(b + l)][2(b - l)])$$

$$\times\{((q^{l+b} + q^{-l-b})(q^{b-l-1} + q^{l-b+1})(q^{l+c'} + q^{-l-c'})(q^{c'-l+1} + q^{l-c'-1}))^{1/2}|l + 1\rangle$$

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$$((q^{l+b-1} + q^{-l-b+1})(q^{b-l} + q^{l-b})(q^{l+c'-1} + q^{-l-c'+1})(q^{c'-l} + q^{l-c'}))^{1/2}|l-1\rangle\}. \qquad (37)$$

At $c' = b+1$ the subspace H_{bc}^- is spanned by the basis vectors $|-b, l\rangle' \equiv |l\rangle$ and the operator $Q_{bc}^-(I_{32})$ is given by the formula similar to the relation (37).

If b and c are as above and $c' = b$, then the subspaces H_{bc}^+ and H_{bc}^- are empty and we have

$$\tilde{T}_{bc}^\epsilon \sim Q_{bc}^0 \oplus Q_{bc}^{++} \oplus Q_{bc}^{--} \quad \text{if} \quad b > 1/2 \quad \text{and} \quad \tilde{T}_{1/2,c}^\epsilon \sim Q_{1/2,c}^{++} \oplus Q_{1/2,c}^{--}.$$

Case 4: Let c be as in Case 2 and let b be of the form $b = b' + \mathrm{i}\pi/2h$, where $b' \in \mathbf{Z}$ if $\epsilon = 0$ and b' is half-integral if $\epsilon = 1$ and such that $b' > c$. In this case the representation space H_ϵ has the irreducible invariant subspaces H_{bc}^0, H_{bc}^+, H_{bc}^{++}, H_{bc}^-, H_{bc}^{--} spanned by all the basis vectors $|k, l\rangle'$ with $-c < l < c$, $c \leq l < b'$, $l \geq b'$, $-b' < l \leq -c$ and $l \leq -b'$, respectively. If $c = 1/2$, then the subspace H_{bc}^0 is empty. We denote the corresponding irreducible subrepresentations of $\tilde{T}_{bc}^\epsilon$ by R_{bc}^0, R_{bc}^+, R_{bc}^{++}, R_{bc}^-, R_{bc}^{--}. We have

$$\tilde{T}_{bc}^\epsilon \sim R_{bc}^0 \oplus R_{bc}^+ \oplus R_{bc}^{++} \oplus R_{bc}^- \oplus R_{bc}^{--} \quad \text{if} \quad c > 1/2,$$

$$\tilde{T}_{b,1/2}^\epsilon \sim R_{b,1/2}^+ \oplus R_{b,1/2}^{++} \oplus R_{b,1/2}^- \oplus R_{b,1/2}^{--}.$$

If $c = 1$, then the subspace H_{bc}^0 is spanned by the vectors $|k, 0\rangle' \equiv |k\rangle$ and the operator $R_{bc}^0(I_{32})$ is given by the formula similar to the relation (36). If $b' = c+1$, then the subspace H_{bc}^+ (the subspace H_{bc}^-) is spanned by the basis vectors $|k, c\rangle' \equiv |k\rangle$ (respectively, by $|k, -c\rangle' \equiv |k\rangle$) and the operator $R_{bc}^+(I_{32})$ (respectively, the operator $R_{bc}^-(I_{32})$) is given by the formula of the type (37).

If b and c are as above and $b' = c$, then the subspaces H_{bc}^+ and H_{bc}^- are empty and we have

$$\tilde{T}_{bc}^\epsilon \sim R_{bc}^0 \oplus R_{bc}^{++} \oplus R_{bc}^{--} \quad \text{if} \quad c > 1/2 \quad \text{and} \quad \tilde{T}_{b,1/2}^\epsilon \sim R_{b,1/2}^{++} \oplus R_{b,1/2}^{--}.$$

Case 5: Let b and c be integral if $\epsilon = 0$ and half-integral if $\epsilon = 1$. (Note that according to our convention we assume that $b \geq 1/2$ and $c \geq 1/2$.) Then the first, second, third and fourth summands on the right hand side of (35) vanish for the appropriate values of k and l. This leads to the decomposition of H_ϵ into the direct sum of the irreducible invariant subspaces $H_{bc}^{\epsilon_1,\epsilon_2}$, $\epsilon_1, \epsilon_2 = 0, +, -$, which are spanned by the basis vectors $|k, l\rangle'$ with

$$-b < k < b, \quad -c < l < c \quad \text{for} \quad E_{bc}^{00}; \qquad -b < k < b, \quad l \geq c \quad \text{for} \quad E_{bc}^{0+};$$

$$-b < k < b, \quad l \leq -c \quad \text{for} \quad E_{bc}^{0-}; \qquad k \geq b, \quad -c < l < c \quad \text{for} \quad E_{bc}^{+0};$$

$$k \geq b, \quad l \geq c \quad \text{for} \quad E_{bc}^{++}; \qquad k \geq b, \quad l \leq -c \quad \text{for} \quad E_{bc}^{+-};$$

$$k \leq -b, \quad -c < l < c \quad \text{for} \quad E_{bc}^{-0}; \qquad k \leq -b, \quad l \geq c \quad \text{for} \quad E_{bc}^{-+};$$

$$k \leq -b, \quad l \leq -c \quad \text{for} \quad E_{bc}^{--}.$$

(Note that the subspaces E_{bc}^{0,ϵ_2} are absent if $c = 1/2$ and the subspaces $E_{bc}^{\epsilon_1,0}$ are absent if $b = 1/2$.) The corresponding subrepresentations of $\tilde{T}_{bc}^\epsilon$ are denoted by $E_{bc}^{\epsilon_1,\epsilon_2}$, respectively. We have

$$\tilde{T}_{bc}^\epsilon = \sum_{\epsilon_1,\epsilon_2=0,+,-} \oplus E_{bc}^{\epsilon_1,\epsilon_2} \quad \text{if} \quad c \neq 1/2, \quad b \neq 1/2,$$

$$\tilde{T}_{1/2,c}^\epsilon = \sum_{\epsilon_1=+,-} \sum_{\epsilon_2=0,+,-} \oplus E_{1/2,c}^{\epsilon_1,\epsilon_2} \quad \text{if} \quad c \neq 1/2,$$

$$\tilde{T}_{b,1/2}^\epsilon = \sum_{\epsilon_1=0,+,-} \sum_{\epsilon_2=+,-} \oplus E_{b,1/2}^{\epsilon_1,\epsilon_2} \quad \text{if} \quad b \neq 1/2,$$

$$\tilde{T}^\epsilon_{1/2,1/2} = E^{++}_{1/2,1/2} \oplus E^{+-}_{1/2,1/2} \oplus E^{-+}_{1/2,1/2} \oplus E^{--}_{1/2,1/2}.$$

Clearly, the irreducible representations E^{00}_{bc} and only they are finite dimensional.

Theorem 2. (1) *The irreducible representations T^ϵ_{bc} and irreducible components of reducible representations T^ϵ_{bc} lead to the following classes of irreducible representations of $U'_q(\mathrm{so}_{2,2})$:*

    (a) *The representations of Theorem 1;*
    (b) *The representations D^0_{bc}, D^+_{bc}, D^-_{bc}, where $b \in \frac{1}{2}\mathbf{Z}$, $b \geq 1/2$;*
    (c) *The representations F^0_{bc}, F^+_{bc}, F^-_{bc}, where $c \in \frac{1}{2}\mathbf{Z}$, $c \geq 1/2$;*
    (d) *The representations Q^0_{bc}, Q^+_{bc}, Q^{++}_{bc}, Q^-_{bc}, Q^{--}_{bc}, where $b \in \frac{1}{2}\mathbf{Z}$ and $c = c' + \mathrm{i}\pi/2h$, $c' \in \frac{1}{2}\mathbf{Z}$, $b, c' \geq 1/2$;*
    (e) *The representations R^0_{bc}, R^+_{bc}, R^{++}_{bc}, R^-_{bc}, R^{--}_{bc}, where $c \in \frac{1}{2}\mathbf{Z}$ and $b = b' + \mathrm{i}\pi/2h$, $b' \in \frac{1}{2}\mathbf{Z}$, $b', c \geq 1/2$;*
    (f) *The representations $E^{\epsilon_1,\epsilon_2}_{bc}$, $\epsilon_1, \epsilon_2 = 0, +, -$, where $b, c \in \frac{1}{2}\mathbf{Z}$, $b, c \geq 1/2$.*

Every irreducible representation of $U'_q(\mathrm{so}_{2,2})$, which is equivalent to some irreducible representation T^ϵ_{bc} or to an irreducible component of some reducible representation T^ϵ_{bc}, is equivalent to one of the representations of classes (a)-(f).

(2) *Between representations of classes (a)-(f) there exist no equivalence relations except for relations which are compositions of the relations (32)-(34).*

Proof. Proof of the assertion (1) is given above. The assertion (2) for the representations of class (a) is given above. Absence of other equivalence relations follows from the fact that representations of any other pair from (a)-(f) have non-coinciding spectra of the operators $T(I_{21}$ and $T(I_{43})$.

8. Irreducible ∗-representations of $U'_q(\mathrm{so}_{2,2})$

The aim of this section is to give the classification of ∗-representations in the set of irreducible representations of $U'_q(\mathrm{so}_{2,2})$ from Theorem 2. This classification is derived by the calculations described, for example, in [15], Sect. 6.4.6. For this reason, we only formulate the final result.

Theorem 3. *The following representations in the set of irreducible representations of Theorem 2 are ∗-representations of $U'_q(\mathrm{so}_{2,2})$:*

    (1) *The representations T^ϵ_{bc} when $b = \mathrm{i}\rho + 1/2$, $c = \mathrm{i}\rho' + 1/2$, $\rho, \rho' \in \mathbf{R}$;*
    (2) *The representations T^ϵ_{bc}, $\epsilon = 0$, when b is in one of the intervals $(0, 1/2]$, $(0 + \mathrm{i}\pi/2h, 1/2 + \mathrm{i}\pi/2h]$ and c is in one of these integrals;*
    (3) *The representations T^ϵ_{bc}, $\epsilon = 0$, when $b = \mathrm{i}\rho + 1/2$, $\rho \in \mathbf{R}$, and c is in one of the intervals $(0, 1/2]$, $(0 + \mathrm{i}\pi/2h, 1/2 + \mathrm{i}\pi/2h]$ or when $c = \mathrm{i}\rho + 1/2$, $\rho \in \mathbf{R}$, and b is in one of these integvals;*
    (4) *The representations D^+_{bc}, D^-_{bc}, $D^0_{1,c}$ when $c = \mathrm{i}\rho + 1/2$, $\rho \in \mathbf{R}$, or when c belongs to one of the intervals $(0, 1/2]$, $(0 + \mathrm{i}\pi/2h, 1/2 + \mathrm{i}\pi/2h]$;*
    (5) *The representations F^+_{bc}, F^-_{bc}, $F^0_{b,1}$ when $b = \mathrm{i}\rho + 1/2$, $\rho \in \mathbf{R}$, or when b belongs to one of the intervals $(0, 1/2]$, $(0 + \mathrm{i}\pi/2h, 1/2 + \mathrm{i}\pi/2h]$;*
    (6) *The representations Q^{++}_{bc}, Q^{--}_{bc}, R^{++}_{bc}, R^{--}_{bc};*
    (7) *The representations Q^+_{bc}, Q^-_{bc} with $c = b + 1 + \mathrm{i}\pi/2h$ and the representations R^+_{bc}, R^-_{bc} with $b = c + 1 + \mathrm{i}\pi/2h$;*
    (8) *The representations Q^0_{bc} with $c = b + \mathrm{i}\pi/2h$ and the representations R^0_{bc} with $b = c + \mathrm{i}\pi/2h$;*
    (9) *The representations $E^{\epsilon_1,\epsilon_2}_{bc}$, $\epsilon_1, \epsilon_2 = +, -$, and the representations $E^{0,\pm}_{1,c}$, $E^{\pm,0}_{b,1}$.*

The representations of class (1) are called ∗-representations of the *principal series*. The representations of class (2) are called ∗-representations of the *supplementary series*. The representations

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of class (3) belong to the mixed (principal–supplementary) series. The representations of $D_{bc}^{\pm}$, $F_{bc}^{\pm}$ of classes (4) and (5) also belong to the mixed (discrete–principal and discrete–supplementary) series. The representations of class (6) are called $*$-representations of the *discrete series*. The representations of $D_{1,c}^{0}$ and $F_{b,1}^{0}$ from classes (4) and (5), the representations of classes (7) and (8) and the representations $E_{1,c}^{0,\pm}$, $E_{b,1}^{\pm,0}$ are called *ladded* $*$-representations. The operators $T(I_{32})$ for these representations are of the types (36) and (37).

9. Conclusions

1. In Theorems 2 and 3 we described sets of irreducible representations and irreducible $*$-representations of the algebra $U_q'(\mathrm{so}_{2,2})$, respectively. The unsolved problem is the following: Do these sets of representations exhaust (up to equivalence) all irreducible representations and $*$-representations of this algebra?

2. The algebra $U_q'(\mathrm{so}_{2,2})$ is a q-deformation of the universal enveloping algebra $U(\mathrm{so}_{2,2})$ of the Lie algebra $\mathrm{so}_{2,2}$. It is well-known that $\mathrm{so}_{2,2}$ is the direct sum of two Lie algebras $\mathrm{so}_{2,1}$: $\mathrm{so}_{2,2} = \mathrm{so}_{2,1} \oplus \mathrm{so}_{2,1}$. Therefore, irreducible representations and irreducible $*$-representations of $\mathrm{so}_{2,2}$ are easily determined by the representations and $*$-representations of the Lie algebra $\mathrm{so}_{2,1}$: any irreducible representation ($*$-representation) T of $\mathrm{so}_{2,2}$ is a direct product of two such representations of $\mathrm{so}_{2,1}$. This trivially gives the classification of irreducible representations and irreducible $*$-representations of $\mathrm{so}_{2,2}$. Comparing these classifications with the irreducible representations of $U_q'(\mathrm{so}_{2,2})$ we see that the representations of classes (d) and (c) have no analogue for the Lie algebra $\mathrm{so}_{2,2}$. Similarly, many $*$-representations of Theorem 3 (for example, the representations with b or c lying in the interval $(0 + \mathrm{i}\pi/2h, 1/2 + \mathrm{i}\pi/2h]$ have no analogue for $\mathrm{so}_{2,2}$.

3. Comparing the classification of irreducible $*$-representations of the q-deformed algebras $U_q'(\mathrm{so}_{2,1})$ and $U_q'(\mathrm{so}_{3,1})$ (see [2]) with irreducible $*$-representations of $U_q'(\mathrm{so}_{2,2})$ in Theorem 3 we make the following conclusions:

(a) The algebra $U_q'(\mathrm{so}_{2,2})$ (unlike the algebras $U_q'(\mathrm{so}_{2,1})$ and $U_q'(\mathrm{so}_{3,1})$) has no strange series of irreducible $*$-representations.

(b) The algebras $U_q'(\mathrm{so}_{2,1})$ and $U_q'(\mathrm{so}_{3,1})$ (unlike the algebra $U_q'(\mathrm{so}_{2,2})$) have no mixed series of irreducible $*$-representations.

These conclusions say that under the transition from the q-deformed algebras corresponding to Lie algebras of rank 1 to the q-deformed algebras corresponding to Lie algebras of higher ranks we obtain a quantitative difference in their representation theories. It is not known now if this property is valid for representations of the Drinfeld-Jimbo algebras.

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